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THE FORMATION OF NON-LINEAR WAVEGUIDES IN THE RESONANT INTERACTION OF THREE SURFACE WAVES†

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The resonant interaction of three two-dimensional wave packets of capillary and flexural–gravitational waves on the surface of an infinitely deep ideal liquid is investigated. Universal asymptotic equations are obtained using the Hamiltonian approach. The main types of decaying interactions are determined as a function of the group velocities of the wave packets. New exact waveguide-type solutions of the equations describing the interaction between three waves are obtained and investigated taking dispersion into account. By reducing the dynamical system to a central manifold it is proved that a wide class of solutions of the waveguide type exists, which depend on five arbitrary constants. It is proved that they are stable with respect to slow changes in the waveguide boundaries and in the amplitudes of the waves propagating along it. © 1997 Elsevier Science Ltd. All rights reserved.

The main feature of the non-linear interaction of three plane wave packets, ignoring dispersion, is the decay of the envelope of the "pumping" wave when it collides with the secondary waves [1, 2]. Complete decay only occurs when the velocity of the "pumping" wave is intermediate. It was assumed in [3] that, in the majority of physical applications, in the resonant interaction of three waves the velocity of the "pumping" wave is extremal. It is shown below that, in the same physical system, different types of decay interactions can occur depending on the relation between the wave vectors of the resonant harmonics.

It was shown in [4] that dispersion effects in the interaction between three plane waves can lead to the formation of coupled solitons, when all three waves propagate under on envelope. Coupled-soliton type solutions were investigated by numerical methods. In this paper new waveguide-type solutions of the equations describing the interaction between three two-dimensional wave packets with dispersion are obtained analytically and investigated.

1. DERIVATION OF THE ASYMPTOTIC EQUATIONS

The equations of the potential motions of an infinitely deep liquid layer can be written in the form [5, 6]

$$\frac{\partial\eta}{\partial t}=\frac{\delta H}{\delta\varphi^s},\quad \frac{\partial\varphi^s}{\partial t}=-\frac{\delta H}{\delta\eta} \tag{1.1}$$

where $z = \eta(t, \mathbf{x})$ and $\varphi^s(t, \mathbf{x})$ are the perturbation of the liquid surface over the horizontal position of equilibrium and the value of the velocity potential on it, $x = (x, y)$ are the horizontal coordinates and z is the vertical coordinate.

The Hamiltonian H is equal to the sum of the total energy of the liquid E and the surface energy F

$$H = E + F \tag{1.2}$$

$$E=\frac{1}{2}\int\limits_{-\infty}^{\infty}\int\left\{\int\limits_{-\infty}^{\eta}\left[(\nabla\varphi)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2\right]dz+g\eta^2\right\}dxdy$$

$$\nabla=\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y}\right)$$

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The Hamiltonian H is a functional of φ^s and η if φ^s is the boundary value of the velocity potential φ, which satisfies Laplace's equation

$$\Delta\varphi = 0, \quad -\infty < z < \eta \tag{1.3}$$

and the boundary condition

$$\partial\varphi / \partial z \to 0, \quad z \to -\infty \tag{1.4}$$

Equations (1.1) are the kinematic boundary condition and the Cauchy–Lagrange integral on the unknown boundary of the liquid $z = \eta$.

The expression for F depends on the physical nature of the phenomena being investigated. We will consider two specific cases below: small-amplitude capillary-gravitational waves and flexural-gravitational waves.

The solutions of Eqs (1.1) for which the condition $\varepsilon = a/\lambda \ll 1$ is satisfied will be called small-amplitude waves, where a and λ are the characteristic amplitude and characteristic width of the wave packet. When investigating the interaction between small-amplitude wave packets the dependence of $(\partial\varphi/\partial z)^s$ on φ^s and η can be obtained in the form of series in powers of ε.

To isolate the parameter ε in (1.1) in explicit form we will change to dimensionless variables, denoted by letters with primes

$$t'\sqrt{\lambda} = t\sqrt{g}, \quad \lambda(x', y', z') = (x, y, z), \quad \varphi'\sqrt{ag\lambda} = \varphi, \quad \eta' a = \eta \tag{1.5}$$

$$E' a^2 g = E, \quad F' a^2 g = F$$

where ρ is the liquid density. The primes will henceforth be omitted.

In dimensionless variables the total energy E, the energy F_{st} of the surface tension [7] and the energy F_{ep} of the elastic plate [8] have the form

$$E = \frac{1}{2}\int\limits_{-\infty}^{\infty}\int \left\{\varphi^s\left[\varepsilon\nabla\varphi^s\nabla\eta - \left(\frac{\partial\varphi}{\partial z}\right)^s(1+\varepsilon^2(\nabla\eta^2))\right] + \eta^2\right\}dxdy$$

$$F_{st} = -\frac{\tau}{2\varepsilon^2}\int\limits_{-\infty}^{\infty}\int \left[1 + \varepsilon^2\left(\frac{\partial\eta}{\partial x}\right)^2 + \varepsilon^2\left(\frac{\partial\eta}{\partial y}\right)^2\right]^{1/2} dxdy$$

$$F_{ep} = \frac{D}{2}\int\limits_{-\infty}^{\infty}\int \left\{(\Delta\eta)^2 + 2(1-\nu)\left[\left(\frac{\partial^2\eta}{\partial x\partial y}\right)^2 - \frac{\partial^2\eta}{\partial x^2}\frac{\partial^2\eta}{\partial y^2}\right]\right\}dxdy$$

$$\tau = \frac{\varkappa}{\rho g\lambda^2}, \quad D = \frac{Eh^3}{12\rho g(1-\nu^2)\lambda^4}$$

Here $\varkappa$ is the surface-tension coefficient, E is Young's modulus, and ν and h are Poisson's ratio and the plate thickness.

The solution of Eq. (1.3) corresponding to fairly rapidly decreasing initial data, has the form

$$\varphi = \int\limits_{-\infty}^{\infty}\int \varphi_f(\mathbf{k}, t)e^{i\mathbf{kx}+kz}d\mathbf{k}, \quad \mathbf{k} = (k_x, k_y), \quad k = |\mathbf{k}| \tag{1.6}$$

We will assume that, at the initial instant of time, the function φ_f is non-zero only in the ε-neighbourhood of the wave vectors $\mathbf{k}_1\mathbf{k}_2$. During the course of time, as a result of non-linear interactions, the spectrum will broaden due to the excitation of multiple harmonics. Hence, at any instant of time we can formally write the expansion

$$\varphi^\circ = \sum_{\mathbf{m}} \overset{\circ}{\varphi}_{\mathbf{m}}(\mathbf{X}, t)e^{i\mathbf{k_m x}}, \quad \overset{\circ}{\varphi}_{\mathbf{m}} = (\overset{\circ}{\varphi}_{-\mathbf{m}})^* \tag{1.7}$$

$$\mathbf{X} = \varepsilon\mathbf{x}, \quad \mathbf{m} = (m_1, m_2), \quad \mathbf{k_m} = m_1\mathbf{k}_1 + m_2\mathbf{k}_2$$

The superscript open circle denotes that the value of the function is taken at $z = 0$, while the asterisk denotes complex conjugation. The symbol $\Sigma_{\mathbf{m}}$ denotes summation over $m_{1,2}$ from $-\infty$ to ∞. It follows from (1.6) and (1.7), to terms of the order of $O(\varepsilon^2)$, that

$$\left(\frac{\partial\varphi}{\partial z}\right)^{\circ} = \sum_{\mathbf{m}}\left\{\left[k - i\varepsilon\nabla_k^{\alpha}k\nabla_{\alpha} - \frac{1}{2}\varepsilon^2\nabla_k^{\alpha\beta}k\nabla_{\alpha\beta}\right]_{\mathbf{m}}\overset{\circ}{\varphi}_{\mathbf{m}}\right\}e^{i\mathbf{k}_{\mathbf{m}}\mathbf{x}} \tag{1.8}$$

$$(\alpha,\beta) = (x,y),\ \ \nabla_k^x = \frac{\partial}{\partial k_x},\ \ \nabla_k^{xx} = \frac{\partial^2}{\partial k_x^2},\ \ \nabla_x = \frac{\partial}{\partial x},\ \ \nabla_{xx} = \frac{\partial^2}{\partial x^2},\ldots$$

The subscript $\mathbf{m}$ denotes that the value of the function is taken when $\mathbf{k} = \mathbf{k}_{\mathbf{m}}$. Summation is assumed over repeated subscripts α, β.

The functions φ^s, $(\partial\varphi/\partial z)^{\circ}$ can be represented, to terms of the order of $O(\varepsilon^2)$, in the form

$$\varphi^s = \left(1 - \frac{\varepsilon^2}{2}\eta^2\Delta\right)\varphi^{\circ} + \varepsilon\eta\left(\frac{\partial\varphi}{\partial z}\right)^{\circ}$$

$$\left(\frac{\partial\varphi}{\partial z}\right)^{s} = \left(1 - \frac{\varepsilon^2}{2}\eta^2\Delta\right)\left(\frac{\partial\varphi}{\partial z}\right)^{\circ} - \varepsilon\eta\Delta\varphi^{\circ} \tag{1.9}$$

From (1.7)–(1.9) we obtain the formulae

$$\left(\eta,\varphi^s,\left(\frac{\partial\varphi}{\partial z}\right)^s\right) = \sum_{\mathbf{m}}(\eta_{\mathbf{m}},\varphi^s_{\mathbf{m}},\varphi_{z,\mathbf{m}})(\mathbf{X},t)e^{i\mathbf{k}_{\mathbf{m}}\mathbf{x}}$$

$$(\overset{*}{\eta}_{-\mathbf{m}},\overset{*}{\varphi^s}_{-\mathbf{m}},\overset{*}{\varphi}_{z,-\mathbf{m}}) = (\eta_{\mathbf{m}},\varphi_{\mathbf{m}},\varphi_{z,\mathbf{m}}) \tag{1.10}$$

$$\varphi_{z,\mathbf{m}} = k_{\mathbf{m}}\varphi^s_{\mathbf{m}} + \varepsilon\sum_{\mathbf{m}}(k^2_{\mathbf{m}-\mathbf{n}} - k_{\mathbf{m}}k_{\mathbf{m}-\mathbf{n}})\eta_{\mathbf{n}}\varphi^s_{\mathbf{m}-\mathbf{n}} - i\varepsilon(\nabla_k^{\alpha}k)_{\mathbf{m}}\nabla_{\alpha}\varphi^s_{\mathbf{m}} + O(\varepsilon^2)$$

Using expansions (1.10), the Hamiltonian H can be expressed explicitly in terms of $\eta_{\mathbf{m}}$, $\varphi^s_{\mathbf{m}}$.

The asymptotic equations for the slowly varying amplitudes of the wave packets have the form [6]

$$\frac{\partial\eta_{\mathbf{m}}}{\partial t} = \frac{\delta\bar{H}}{\delta\varphi^s_{-\mathbf{m}}},\quad \frac{\partial\varphi^s_{\mathbf{m}}}{\partial t} = -\frac{\delta\bar{H}}{\delta\eta^s_{-\mathbf{m}}} \tag{1.11}$$

An expression for the average Hamiltonian $\bar{H}$ can be obtained from (1.2) and (1.10) by dropping all terms under the integral proportional to the rapidly varying exponential function.

Substituting the following formulae into (1.11)

$$\eta_{\mathbf{m}}(\mathbf{X},t) = \tilde{\eta}_{\mathbf{m}}(\mathbf{X},T)e^{i\omega_{\mathbf{m}}t},\ \ \varphi^s_{\mathbf{m}}(\mathbf{X},t) = \varphi_{\mathbf{m}}(\mathbf{X},T)e^{i\omega_{\mathbf{m}}t},\ \ \omega_{\mathbf{m}} = m_1\omega_1 + m_2\omega_2,\ \ T = \varepsilon t$$

$$\omega_{\mathbf{m}} = \omega(\mathbf{k}_{\mathbf{m}}) = k_{\mathbf{m}}L(\mathbf{k}_{\mathbf{m}}) \tag{1.12}$$

where $\omega = \omega(k)$ is the dispersion relation of the linear approximation, and eliminating $\tilde{\eta}_{\mathbf{m}}$ from the equations obtained, we have

$$i\frac{\omega^2_{\mathbf{m}} - \omega^2(\mathbf{k}_{\mathbf{m}})}{\omega_{\mathbf{m}}}\varphi_{\mathbf{m}} + 2\varepsilon\left(\frac{\partial}{\partial T} - \nabla_k^{\alpha}\omega_{\mathbf{m}}\nabla_{\alpha}\right)\varphi_{\mathbf{m}} +$$

$$+\varepsilon\sum_{\mathbf{m}}\alpha_{\mathbf{mn}}\varphi_{\mathbf{m}-\mathbf{n}} + i\varepsilon^2\nabla^{\alpha\beta}\omega_{\mathbf{m}}\nabla_{\alpha\beta}\varphi_{\mathbf{m}} + \ldots = 0 \tag{1.13}$$

$$\alpha_{\mathbf{mn}} = \frac{L_{\mathbf{m}}k_{\mathbf{n}}}{\omega_{\mathbf{m}}\omega_{\mathbf{n}}}(k_{\mathbf{m}}k_{\mathbf{m}-\mathbf{n}} - k^2_{\mathbf{m}-\mathbf{n}} - \mathbf{k}_{\mathbf{n}}\mathbf{k}_{\mathbf{m}-\mathbf{n}}) - \frac{1}{2}(\mathbf{k}_{\mathbf{n}}\mathbf{k}_{\mathbf{m}-\mathbf{n}} + k_{\mathbf{n}}k_{\mathbf{m}-\mathbf{n}})$$

It can be seen from (1.13) that the amplitude $\varphi_{\mathbf{m}}$ is of the order of ε if $\omega^2_{\mathbf{m}} - \omega^2(\mathbf{k}_{\mathbf{m}}) = O(1)$. Hence, harmonics whose wave numbers satisfy the synchronism conditions

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3, \quad \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) = \omega(\mathbf{k}_3) \tag{1.14}$$

interact most intensely.

The dispersionless equations of the resonant interaction of three wave packets follow from (1.13)

$$i\left(\frac{\partial}{\partial T} - \nabla_k^\alpha \omega_j \nabla_\alpha\right)\varphi_j = W c_j N_j, \quad j = 1,2,3 \tag{1.15}$$

$$N_1 = \overset{*}{\varphi}_2 \varphi_3, \quad N_2 = \overset{*}{\varphi}_1 \varphi_3, \quad N_3 = \varphi_1 \varphi_2, \quad c_j = \omega_j / k_j$$

$$W = \frac{1}{2}[c_2^{-1}(k_1 k_3 - \mathbf{k}_1 \mathbf{k}_3) + c_1^{-1}(k_2 k_3 - \mathbf{k}_2 \mathbf{k}_3) - c_3^{-1}(k_1 k_2 + \mathbf{k}_1 \mathbf{k}_2)]$$

Equations (1.15) are universal and, when considering different surface phenomena, only the coefficients for the non-linearity will differ, and these are determined by the dispersion relation of the linear approximation.

The properties of the solutions of Eqs (1.15) depend on the relations between the group velocities and on the signs of the coefficients of the quadratic terms, which are the same in the case considered. Hence, it follows that it is impossible to obtain explosive-type interactions [3].

In the general case of the interaction of two-dimensional wave packets with unequal group velocities, solutions describing partial or complete decay (depending on the ratio of the amplitudes) of a wave φ_3 localized in a plane to localized waves φ_1 and φ_2 are obtained analytically. Here the form of the wave packets does not change [9].

2. THE CONDITIONS FOR RESONANT INTERACTION

We will assume that there is a direction in the (x, y) plane along which the group velocities of the interacting wave packets $\nabla_k^x \omega_j$ are equal to c. We will choose this direction as the x axis. Changing to a system of coordinates which moves in space with velocity c, we obtain that, in Eqs (1.15), the partial derivatives with respect to the x direction disappear, and the system reduces to the plane case.

The properties of the solutions of system (1.15) in the plane case have been investigated in detail in [3, 10, 11] and depend on the ratio of the group velocities $\nabla_k^y \omega_j$. If the velocity $\nabla_k^x \omega_3$ of the pumping wave φ_3 is intermediate, $\nabla_k^x \omega_3 \in (\nabla_k^y \omega_1, \nabla_k^y \omega_2)$, then φ_3 decays into φ_1, φ_2 completely over an infinite time interval for values of φ_1, φ_2 as small as desired until interaction begins. The solutions of the equations describing this phenomenon can be obtained analytically using the inverse scattering problem method [1, 3] or by a shift in symmetry group from the zero "bare" solution [10, 11].

It was pointed out in [3] that, in the majority of problems having a physical application, the velocity of the pumping wave turns out to be extremal. In this case the analytically obtained solutions have no physical meaning since they contain singularities where the amplitudes φ_j become infinite at some instant of time. Numerical investigations [1, 2] show that the intensity of the pumping wave falls considerably only after collision with the fairly large wave φ_1 or φ_2. Here, as a result of interaction, a considerable portion of the energy transfers into ripple waves, which form a "tail" on the soliton of the pumping wave.

In both cases, the decay processes depend very much on the amplitudes of the interacting waves. If φ_j depends on x, the interaction parameters will be different for different values of x. Hence it can be seen that the derivatives $\partial\varphi_j/\partial x$ may increase with time. In this case dispersion in the x direction may influence the interacting wave packets.

The equations of the interaction of three waves, taking dispersion in the x direction into account, can be derived from (1.13) in the same way as (1.15). Here it is assumed that the characteristic dimensions of the wave packets are such that the condition $\partial\varphi_j/\partial Y = O(\partial^2\varphi_j/\partial X^2)$ is satisfied. In a system of coordinates moving along the x axis with velocity c, they have the form

$$i\left(\frac{\partial}{\partial T} - \nabla_k^y \omega_j \frac{\partial}{\partial Y}\right)\varphi_j + \frac{1}{2}\nabla_k^{xx}\omega_j \frac{\partial^2}{\partial \xi^2}\varphi_j = W c_j N_j, j = 1,2,3 \tag{2.1}$$

$$\xi = X + cT$$

Note that the equations of three-wave interaction, taking dispersion into account, were investigated previously in [4, 11]. The additional dispersion term in the equations in this case is much less than the terms with first derivatives, since it is proportional to $\partial^2\varphi_j/\partial Y^2$. Hence, when investigating the exact solutions one must ensure that this condition is satisfied. In the case considered all the terms in (2.1) have the same order of smallness.

We will consider the conditions for deriving Eqs (2.1) in more detail. They consist of the synchronism conditions (1.14) and the conditions for the group velocities to be equal along the x axis

$$\nabla_k^x\omega_1 = \nabla_k^x\omega_2 = \nabla_k^x\omega_3 \tag{2.2}$$

For convenience we will take the solutions of Eqs (1.14) in the form

$$\begin{aligned} &\mathbf{k}_1 = k(\cos\theta_1, \sin\theta_1),\ \mathbf{k}_2 = \alpha k(\cos\theta_2, \sin\theta_2) \\ &\mathbf{k}_3 = k(\cos\theta_1 + \alpha\cos\theta_2,\ \sin\theta_1 + \alpha\sin\theta_2) \end{aligned} \tag{2.3}$$

Substituting (2.3) into (1.14) we obtain the following equations for capillary–gravitational waves and flexural–gravitational waves, respectively

$$\tau k^2 = r_{st}(\theta, \alpha),\ Dk^4 = r_{ep}(\theta, \alpha),\ \theta = \theta_1 - \theta_2 \tag{2.4}$$

Here

$$\begin{aligned} &r_{st,ep} = \left(b_{st,ep} + \sqrt{b_{st,ep}^2 - f_{st,ep}g_{st,ep}}\right) / f_{st,ep} \\ &b_{st} = 2\alpha(1+\alpha^2) - A_1A_3,\ b_{ep} = 2\alpha(1+\alpha^4) - A_1A_5 \\ &f_{st} = A_3^2 - 4\alpha^3,\ f_{ep} = A_5^2 - 4\alpha^5,\ g_{st,ep} = A_1^2 - 4\alpha \\ &\alpha = \sqrt{1+\alpha^2+2\alpha\cos\theta},\ A_n = 1 + \alpha^n - \bar{a}^n \end{aligned}$$

Formulae (2.4) parametrically define the wave vectors of the interacting wave packets. The parameters are the ratio of the moduli of the wave vectors $\alpha = k_2/k_1$ and the angle θ between them.

The functions r_{st} and r_{ep} have a physical meaning only in regions where they are greater than zero. In Figs 1 and 2 these regions are bounded by the straight lines $\alpha = 0, \theta = 0, \theta = \pi$ and the curves A_1B_1, B_2C_1, A_2C_2 for r_{st} and AB, BC and AC for r_{ep}. The functions r_{st} and r_{ep} tend to infinity as one approaches $\alpha = 0$ and the curvilinear boundaries.

Equations (2.2) have the trivial solution $\theta_1 = \theta_2 = \pi/2$. In this case the conditions $\nabla_k^y\omega_i = 0$ and $\nabla_k^y\omega_3 > (\nabla_k^y\omega_1, \nabla_k^y\omega_2)$ are satisfied, i.e. the velocity of the pumping wave is extremal.

A non-trivial solution exists for flexural-gravitational waves. It is represented in Fig. 2 by the curves EF and IH. The boundary points of these curves have coordinates E(0.5, π), F(0.88, 0.8π), I(1.14, 0.8π), H(1.47, π). It can be shown that for this solution the following conditions are satisfied

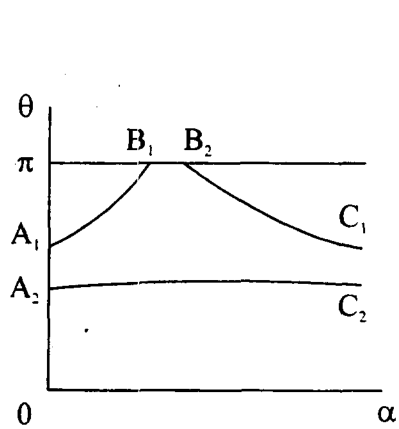

Fig. 1.

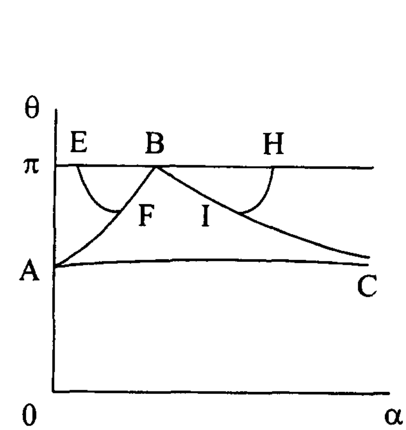

Fig. 2.

$$\nabla_k^y \omega_1 > \nabla_k^y \omega_3 > \nabla_k^y \omega_2, \; \nabla_k^y \omega_2 < 0, \; \theta_1 \in (0, \pi/2)$$

i.e. the velocity of the pumping wave is intermediate.

Hence, it has been shown that different types of decaying interactions, described by system (2.1), can exist in one and the same system depending on the wave vectors of the interacting waves.

3. THE EXISTENCE OF WAVEGUIDE-TYPE SOLUTIONS

We will investigate some properties of Eqs (2.1). This system has two conservation laws which are an analogue of the Manley–Rowe relations, and the law of conservation of energy

$$\frac{\partial}{\partial T}(c_{1,2}^{-1}|\varphi_{1,2}|^2 + c_3^{-1}|\varphi_3|^2) + \frac{\partial}{\partial Y}(c_{1,2}^{-1}\nabla_k^y\omega_{1,2}|\varphi_{1,2}|^2 + c_3^{-1}\nabla_k^y\omega_3|\varphi_3|^2) +$$
$$+\frac{\partial}{\partial \xi}\operatorname{Im}[c_{1,2}^{-1}\nabla_k^{xx}\omega_{1,2}\varphi_{1,2}^*\frac{\partial}{\partial \xi}\varphi_{1,2} + c_3^{-1}\nabla_k^{xx}\omega_3\varphi_3^*\frac{\partial}{\partial \xi}\varphi_3] = 0 \tag{3.1}$$

$$\frac{\partial}{\partial T}\int\limits_{-\infty}^{\infty}\int\left[\sum_{j=1}^{3}c_k c_l\left(\nabla_k^y\omega_j \operatorname{Im}\left(\varphi_j^*\frac{\partial}{\partial Y}\varphi_j\right) + \frac{1}{2}\nabla_k^{xx}\omega_j\left|\frac{\partial}{\partial \xi}\varphi_j\right|^2\right)+\right.$$
$$\left. + 2Wc_1c_2c_3\operatorname{Re}(\varphi_1\varphi_2\varphi_3^*)\right]d\xi dY = 0 \quad (j,k,l) = (1,2,3) \tag{3.2}$$

We will seek a solution of (2.1) in the form

$$\varphi_j = \varphi_j(\zeta)\exp i(r_j^x\xi + r_j^y Y + s_j T) \tag{3.3}$$
$$\zeta = c_x\xi + c_y Y + T, \; r_1^{x,y} + r_2^{x,y} = r_3^{x,y}, \; s_1 + s_2 = s_3$$

Assuming that

$$c_y = \Delta^{-1}(\nabla_k^{xx}\omega_3(\nabla_k^{xx}\omega_1 + \nabla_k^{xx}\omega_2) - \nabla_k^{xx}\omega_1\nabla_k^{xx}\omega_2)$$
$$r_{1,2}^{x,y} = (c_x\Delta)^{-1}(\nabla_k^{xx}\omega_3(\nabla_k^y\omega_{1,2} - \nabla_k^y\omega_{2,1}) + \nabla_k^{xx}\omega_{2,1}(\nabla_k^y\omega_3 - \nabla_k^y\omega_{1,2}) \tag{3.4}$$
$$\Delta = \nabla_k^{xx}\omega_3(\nabla_k^y\omega_1\nabla_k^{xx}\omega_2 + \nabla_k^y\omega_2\nabla_k^{xx}\omega_1) - \nabla_k^y\omega_3\nabla_k^{xx}\omega_1\nabla_k^{xx}\omega_2$$

substituting (3.3) into (2.1) and making a scale transformation of φ_j, we obtain

$$\ddot{\varphi}_j - \Delta_j\varphi_j = \varphi_k\varphi_l, \quad (j,k,l) = (1,2,3) \tag{3.5}$$
$$\Delta_j = (2s_j + 2\nabla_k^y\omega_j^y + \nabla_j^{xx}(r_j^x)^2)/(c_x^2\nabla_k^{xx}\omega_j)$$

The system of equations (3.5) has the integral

$$\sum_{j=1}^{3}(\Delta_j\varphi_j^2 - \dot{\varphi}_j^2) + 2\varphi_1\varphi_2\varphi_3 = C, \quad C = \text{const} \tag{3.6}$$

Note that the three quantities Δ_j depend on five arbitrary constants c_x, $r_{1,2}^y$, $s_{1,2}$. Hence they can always be chosen so that all Δ_j will be the same and equal to $\Delta > 0$. In this case it is easy to find a class of exact solutions of (3.5) which are expressed in terms of elliptic functions [14, 15]; among these solutions there are localized waveguide-type solutions

$$\varphi_j^* = -\frac{3}{2}\operatorname{ch}^{-2}\left(\frac{\sqrt{\Delta}}{2}\zeta\right) \tag{3.7}$$

The parameters c_x, $r^y_{1,2}$, $s_{1,2}$ in solutions (3.7) are not arbitrary and are connected by relations that arise from the equality $\Delta_1 = \Delta_2 = \Delta_3 > 0$. Some particular solutions of system (3.5) for other values of these parameters were obtained numerically in [15]. We will show that for a sufficiently small change in the parameters Δ_j in the neighbourhood of Δ, Eqs (3.5) admit of localized solutions of the type (3.7). It is obvious that this family of solutions will be a five-parameter solution with parameters c_x, $r^y_{1,2}$, $s_{1,2}$.
The last assertion will be formulated in the form of the following theorem.

Theorem 1. We will assume that $\varphi^* = (\varphi_1^*, \varphi_2^*, \varphi_3^*)^t$, $\varphi_1^* = \varphi_2^* = \varphi_3^* = \varphi^*$ is a soliton-like solution of system (3.5) for $\Delta_i = \Delta$ $(k = 1, 2, 3)$. Then, for sufficiently small μ_i a family of soliton-like solutions $\varphi = (\varphi_1, \varphi_2, \varphi_3)^t$ of Eqs (3.5) exists with $\Delta_i = \Delta + \mu$. In addition, the following limits hold

$$|\boldsymbol{\varphi} - \boldsymbol{\varphi}^*| \leqslant c_0 |\boldsymbol{\mu}| \exp(-\sigma|\zeta|)$$

where $\mu = (\mu_1, \mu_2, \mu_3)^t$, $\sigma < \sqrt{\Delta}$ and c_0 is independent of μ.

Proof. We represent the solution of Eqs (3.5) for Δ_j, indicated in the formulation of the theorem, in the form $\varphi_i = \varphi^* + \hat{\varphi}_i$. Then, system (3.5) can be written in the following matrix form

$$L\boldsymbol{\varphi} = \mathbf{g} \tag{3.8}$$

$$L = \begin{Vmatrix} \dfrac{d^2}{dx^2} - \Delta & -\varphi^* & -\varphi^* \\ -\varphi^* & \dfrac{d^2}{dx^2} - \Delta & -\varphi^* \\ -\varphi^* & -\varphi^* & \dfrac{d^2}{dx^2} - \Delta \end{Vmatrix}$$

$$g_i = \mu_i \hat{\varphi}_i + \mu_i \varphi^* + \hat{\varphi}_k \hat{\varphi}_l, \quad i \neq k \neq l$$

We further define the Banach spaces

$$C^e_{\sigma,j} = \left\{ \varphi \in C^j(\mathbf{R}), \ \sup_x \exp(\sigma|\zeta|)|\varphi^{(m)}| < \infty, \ \varphi(\zeta) = \varphi(-\zeta), \ j = 0, 2, \ m \leqslant j \right\}$$

and we write $X_\sigma = C^e_{\sigma,2} \times C^e_{\sigma,2} \times C^e_{\sigma,2}$ and $Y_\sigma = C^e_{\sigma,0} \times C^e_{\sigma,0} \times C^e_{\sigma,0}$. It is obvious that $\varphi^* \in X_{\sqrt{\Delta}}$.
The existence and uniqueness of $\varphi(\mu) \in X_\sigma$ (for sufficiently small μ), follows from the theorem of implicit functions; these satisfy system (3.5) if the operator $L: X_\sigma \to Y_\sigma$ has a bounded inverse. In other words, it is required to prove that the equation

$$L\varphi = \mathbf{f} \tag{3.9}$$

has a unique solution $\varphi \in X_\sigma$ for each $\mathbf{f} \in Y_\sigma$. System (3.9) is equivalent to the system obtained from it by pairwise subtraction of the equations. The latter, in turn, is equivalent to the equation

$$l\chi + \varphi^* \chi = f (l = d^2/dx^2 - \Delta: \ C^e_{\sigma,2} \to C^e_{\sigma,0}), \tag{3.10}$$

where χ is the difference between $\hat{\varphi}_i$ and $\hat{\varphi}_j$ with any unequal subscripts i and j, and $f \in C^e_{\sigma,0}$. Hence, the proof of the invertibility of the operator L reduces to proving that Eq. (3.10) has a unique solution for any f from $C^e_{\sigma,0}$.
The operator l is invertible since $\Delta > 0$. Hence, (3.10) can be rewritten in the form

$$(1 + l^{-1}\varphi^*)\chi = l^{-1}f$$

The operator $l^{-1}\varphi^*$: $C^e_{\sigma,2} \to C^e_{\sigma,2}$, $\sigma < \sqrt{\Delta}$ is compact. Indeed, the set bounded in $C^e_{\sigma,2}$ is mapped to this operator in the bounded set in $C^e_{\sigma,2}$ with elements having a uniform exponential decrease at infinity. This indicates that the Artsel–Ascoli theorem is applicable and the transform of the bounded set is precompact. We will show that 1 is not an eigenvalue of the operator $-l^{-1}\varphi^*$. It then follows from Theorem 1 on page 476 of [12] that the inverse operator $(1 + l^{-1}\varphi^*)^{-1}$ is invertible and continuous. The equation $l^{-1}\varphi^*\chi = \chi$ is equivalent to Schrödinger's equation with a negative scattering potential $\chi'' - \Delta\chi + \varphi^*\chi = 0$, and of course, the corresponding Sturm–Liouville operator does not have a discrete spectrum. Hence, it follows that $\chi = 0$ is a unique solution belonging to $C^e_{\sigma,2}$.

We will write (3.8) in the form

$$L\hat{\varphi} + G(\hat{\varphi}) = \psi, \quad \|G\|_{Y_\sigma} \leq \gamma\|\hat{\varphi}\|^2_{X_\sigma}\|\psi\|_{X_\sigma} \leq c|\mu| \tag{3.11}$$

where $\|\cdot\|_{X_\sigma}$ and $\|\cdot\|_{Y_\sigma}$ are norms in the corresponding spaces. The bounds indicated in the formulation of the theorem then quickly follow from (3.11).

4. ASYMPTOTIC FORMULAE FOR NON-LINEAR WAVEGUIDES

We will demonstrate the existence of waveguide-type solutions of system (3.5) which differ from (3.7). To do this we will use the method of reducing the dynamical system generated by (3.5) to a central manifold [13, 14]. The flow on the central manifold describes all the bounded solutions which do not leave a small neighbourhood of the state of rest for all positive and negative values of the "time". The spatial variable $-\infty < x < \infty$ plays the part of "time" for travelling waves. All the small bounded solutions of the dynamical system

$$\dot{\mathbf{w}} = A\mathbf{w} + \mathbf{F}(\mu, \mathbf{w}) \tag{4.1}$$

($A : \mathbf{R}^n \to \mathbf{R}^n$ is a linear operator—a constant matrix, among the eigenvalues of which there are pure imaginary ones, $\mathbf{F} \in C^k(\mathbf{R}^n, \mathbf{R}^n)$ for fairly large integer k, $\mathbf{F}(0, 0) = 0$, $\partial_{\mathbf{w}}\mathbf{F}(0, 0) = 0$ and μ is the bifurcation parameter) lie on the central manifold $\mathbf{w}_1 = \mathbf{h}(\mu, \mathbf{w}_0)$, $\mathbf{w}_0 \in E_0$, invariant to the space extended on the eigenvectors corresponding to pure imaginary eigenvalues A, $\mathbf{w}_1 \in E_h$, $\mathbf{R}^n = E_0 \oplus E_h$ and $\mathbf{w} = \mathbf{w}_0 + \mathbf{w}_1$. The function $\mathbf{h}$ has second order of smallness with respect to μ and $\mathbf{w}_0$ and inherits the properties of symmetry of system (4.1).

We will distinguish [15] three successive stages of the investigation of the existence of soliton-like solutions in the case considered:

—reduction of the sixth-order dynamic system (3.5) to a second-order system describing the flux on the central manifold;

—the approximation of the system on the central manifold to a sequence of integrable systems (in quasi-normal form [14]) and the description of the soliton-like solutions of the system in quasi-normal form;

—proof of the roughness of the soliton-like solutions with respect to any inverse perturbations of higher order in amplitude.

The invertibility of these perturbations satisfies the corresponding symmetry of the initial equations. The sequence of solutions of the systems of equations in quasi-normal form is therefore as accurate an approximation as desired of the soliton-like solution of the complete system (3.5).

We will put $\varphi_1 = \varphi_3$, $\Delta_1 = \Delta_3 = \mu$ in (3.5), where μ is a small quantity, and $\Delta_2 = \Delta > 0$. In this case system (3.5) can be represented in the form (4.1) where $\mathbf{w} = (\varphi_1, \varphi_2, \eta_1, \eta_2)^t$, $\eta_j = \varphi_j'$, $j = 0, 1$

$$A = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & \Delta & 0 & 0 \end{Vmatrix}, \quad \mathbf{F} = \begin{Vmatrix} 0 \\ 0 \\ \mu\varphi_1 + \varphi_1\varphi_2 \\ \varphi_1^2 \end{Vmatrix} \tag{4.2}$$

Note that the right-hand side of (3.5) in the case considered anticommutes with isometry $R = \mathrm{diag}(1, 1, -1, -1)$, i.e. $AR = -RA$ and $\mathbf{F}(\mu, R\mathbf{w}) = -R\mathbf{F}(\mu, \mathbf{w})$. This indicates that system (3.5) is invertible, i.e. among its solutions there are solutions with even φ_1 and φ_2 and odd η_1 and η_2. These solutions will also be called invertible solutions. The eigenvalues λ of matrix A satisfy the equation $\lambda^2(\lambda^2 - \Delta) = 0$, which has four roots, and of those of zero multiplicity two lie on the imaginary axis. Note that $|\mathbf{F}(\mu, \mathbf{w})| \leq c(|\mu||\mathbf{w}| + |\mathbf{w}|^2)$ for $\mathbf{w}$ sufficiently close to zero.

Suppose $\mathbf{w} = \mathbf{w}_0 + \mathbf{w}_1$, where $\mathbf{w}_0 \in E_0$, while $\mathbf{w}_1 \in E_h$, and similarly for $\mathbf{F}_0$ and $\mathbf{F}_1$; $A_0 = A|_{E_0}$, $A_1 = A|_{Eh}$, and the system of equations (3.5) has the form

$$\dot{\mathbf{w}}_0 = A_0\mathbf{w}_0 + \mathbf{F}_0(\mu,\mathbf{w}_0 + \mathbf{w}_1),\ \dot{\mathbf{w}}_1 = A_1\mathbf{w}_1 + \mathbf{F}_1(\mu, \mathbf{w}_0 + \mathbf{w}_1) \tag{4.3}$$

In addition, neighbourhoods of zero $U_0' \subset E_0$, $U_1' \subset E_h$ and a neighbourhood Λ of the point $\mu = 0$ obviously exist such that

$$F = (\mathbf{F}_0, \mathbf{F}_1)' \in C^k(\Lambda \times U_0' \times U_1', E_0 \times E_h)$$
$$F(0, 0),\ \partial_w \mathrm{F}(0, 0) = 0$$

We further use the theorem on a central manifold [14], according to which neighbourhoods of zero $U_0 \subset U_0' \subset E_0$, $U_1 \subset U_1' \subset E_h$ a neighbourhood $\Lambda_0 \subset \Lambda$ of the point $\mu = 0$ and a function

$$\mathbf{h}(\mu, \mathbf{w}_0) \in C^{k-1}(\Lambda_0 \times U_0, U_1)$$

exist which possess the following properties:
—the set

$$M = \{(\mathbf{w}_0, \mathbf{h}(\mu, \mathbf{w}_0) \subset E_0 \times E_h | \mathbf{w}_0 \in U_0\}$$

is a local integral manifold of system (4.3) with $\mu \in \Lambda_0$;
—each solution of system (4.3) with $\mu \in \Lambda_0$, $\mathbf{w}_0(\xi)$, $\mathbf{w}_1(\xi) \in U_0 \times U_1$ for all $\xi \in \mathbf{R}$ belongs to M;

$$\mathbf{h}(0, 0) = \partial_{\mathbf{w}_0}\mathbf{h}(0, 0) = 0;$$

if R_0: $E_0 \to E_0$, R_1: $E_h \to E_h$ are linear isometries, so that $\mathbf{F}_j(\mu, R_0\mathbf{w}_0 R_1\mathbf{w}_1) = -R_j\mathbf{F}_j(\mu, \mathbf{w}_0, \mathbf{w}_1)$, $A_jR_j = -R_jA_j$ $(j = 0, 1)$, then $\mathbf{h}(\mu, R_0\mathbf{w}_0) = R_1\mathbf{h}(\mu, \mathbf{w}_1)$.

From the theorem on linear manifolds there follows the reduction of system (4.3) for the solutions which remain in $U_0 \times U_1$ for all $x \in \mathbf{R}$. These solutions satisfy the equations

$$\dot{\mathbf{w}}_0 = A_0\mathbf{w}_0 + \mathbf{f}_0(\mu, \mathbf{w}_0),\ \mathbf{f}_0(\mu, \mathbf{w}_0) = \mathbf{F}_0(\mu, \mathbf{w}_0 + \mathbf{h}(\mu, \mathbf{w}_0)) \tag{4.4}$$

Moreover, the solutions of Eqs (4.4) are invertible with respect to the matrix R_0: $R = R_0 \oplus R_1$, R_0: $E_0 \to E_0$. In the case considered the dimension of E_0 (and, of course, the central manifold also) is equal to two. The eigenvector and associated vector of the operator A corresponding to $\lambda = 0$, have the form

$$\varphi_0 = \mathrm{col}(1, 0, 0, 0),\ \varphi_1 = \mathrm{col}(0, 0, 1, 0)$$
$$(A_0\varphi_0 = 0,\ A_0\varphi_1 = \varphi_0,\ R_0\varphi_0 = \varphi_0,\ R_0\varphi_1 = -\varphi_1)$$

and of course

$$A_0 = \begin{Vmatrix} 0 & 1 \\ 0 & 1 \end{Vmatrix},\quad R_0 = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}$$

For $\mathbf{w}_0 \in E_0$ we have

$$\mathbf{w}_0 = \mathrm{col}(\varphi_1, \varphi_2, \eta_1, \eta_2) = a_0\varphi_0 + a_1\varphi_1$$

The associated vector ψ_0 and the eigenvector ψ_1 of the operator A^*, conjugate to the operator A, are identical with φ_0 and φ_1, respectively.

Equation (4.4) in the φ_0, φ_1 basis has the form

$$\dot{\mathbf{a}} = A_0\mathbf{a} + \mathbf{g}_0(\mu, \mathbf{a}),\ \mathbf{a} = (a_0, a_1)' \tag{4.5}$$

and the vector $\mathbf{g}_0 = O(|\mu| \mathbf{a}| + |\mathbf{a}|^2)$ has the components $\langle \mathbf{F}, \varphi_j \rangle$ $(j = 0, 1)$ $\mathbf{F}$ is the vector (4.2) and $\langle\ ,\ \rangle$ denotes the scalar products in R^4, where

$$\langle \mathbf{F}, \varphi_0 \rangle = 0\ \langle F, \varphi_1 \rangle = 1/2(\mu a_0 + a_0\varphi_2)$$

The functions φ_2 and η_2 are of the second order of smallness in $\mathbf{a}$ and can be expressed in terms of the vector function $\mathbf{h}(\mu, \mathbf{a}) = (h_0, h_1)^t$, which occurs in the theorem on the central manifold, from the formula $\varphi_2 = h_0 + h_1$.

We will further calculate the components h_0 and h_1, apart from terms of the order of $\mu|\mathbf{a}|^2 + |\mathbf{a}|^2$, which satisfy the second system of equations of (4.3), from which we have in this case

$$\varphi_2 = -\frac{1}{\Delta}a_0^2 - \frac{2}{\Delta^2}a_1^2 + o(\mu|\mathbf{a}|+|\mathbf{a}|^2)$$

Hence it follows that the flux on the central manifold has the form

$$\dot{a}_0 = a_1,\ \dot{a}_1 = \frac{1}{2}\mu a_0 - \frac{1}{2\Delta}a_0^3 - \frac{1}{\Delta^2}a_0 a_1^2 + o(\mu|\mathbf{a}|+|\mathbf{a}^3|) \tag{4.6}$$

Carrying out the scaling $\alpha_0 = 2\Delta|\mu|^{1/2}\beta_0(\xi)$, $\alpha_1 = 2\Delta|\mu|^{1/2}\delta\beta_1(\xi)$, $\xi = \delta\zeta$, $\delta = |\mu/2|^{1/2}$ we obtain, up to terms of the order of $O(\mu)$

$$\partial_\xi\beta_0 = \beta_1,\ \partial_\xi\beta_1 = \operatorname{sign}(\mu)\beta_0 - \beta_0^3 \tag{4.7}$$

Equations (4.7) for $\mu > 0$ have solitary-wave type solutions $\beta_0 = \pm\sqrt{(2)}\,\mathrm{ch}^{-1}\,\xi/2$.

Theorem 2 (*on the roughness of soliton-like solutions*). We will assume that $\beta^* = (\beta_0^*, \beta_1^*)$ is a soliton-like solution of Eqs (4.7). Then, for sufficiently small μ_0, $\mu \in (0, \mu_0)$ a family of soliton-like solution $\alpha = (\alpha_0, \alpha_1)(\mu)$ of the complete system (4.7) exists. The following limits hold

$$|\alpha - \alpha^*| \leqslant c_0\mu\exp(-\sigma|\zeta|)$$

where c_0 depends only on μ_0, $\sigma < 1$.

Theorem 2 is proved in the same way as Theorem 1. The solutions considered therefore have the form

$$\varphi_1^* = \varphi_3^* = \pm\frac{\sqrt{2}\Delta|\mu|^{1/2}}{\mathrm{ch}(|\mu/2|^{1/2}\zeta)} + O(\mu),\ \varphi_2^* = -\frac{2\Delta|\mu|}{\mathrm{ch}^2(|\mu/2|^{1/2}\zeta)} + o(\mu)$$

5. STABILITY OF NON-LINEAR WAVEGUIDES

In Sections 3 and 4 we proved the existence of five-parameter families of waveguide-type solutions of Eqs (3.5). We will investigate their evolution for slow localized perturbations of the wavefront, amplitude and phase of the waves propagating in the waveguide

$$\begin{gathered}\varphi_j = \varphi_j(c_x(\mu Y, \mu T)\xi + c_y Y + T)\exp i(r_j^x\xi + r_j^y Y + s_j T + \theta_j(\mu Y, \mu T))\\ r_1^x = R_1/c_x,\ r_2^x = R_2/c_x,\ r_3^x = r_1^x + r_2^x,\ \theta_3 = \theta_1 + \theta_2,\ \mu \ll 1\end{gathered} \tag{5.1}$$

A slow change in c_x denotes a small change in the wavefront and amplitude of the soliton φ_j. The functions $\theta_j(\mu Y, \mu T)$ correspond to a slowly varying wave phase. When $\mu = 0$ the functions (5.1) satisfy Eqs (3.5).

Substituting (5.1) into (2.1), we obtain, to the first order in μ

$$\begin{gathered}\frac{\partial\varphi_j}{\partial\zeta}\xi\left(\frac{\partial}{\partial\tilde{T}} + \nabla_k^y\omega_j\frac{\partial}{\partial\tilde{Y}}\right)c_x + \varphi_j\left(\frac{\partial}{\partial\tilde{T}} + \nabla_k^y\omega_j\frac{\partial}{\partial\tilde{Y}}\right)\theta_j = 0\\ \tilde{T} = \mu T,\ \tilde{Y} = \mu Y\end{gathered}$$

Integrating these equations with respect to ξ from $-\infty$ to ∞, we obtain a system, the general solution of which has the form

$$\begin{gathered}c_x = \exp(f_1 + f_2 - f_3),\ f_j = f_j(\tilde{Y} - \nabla_k^y\omega_j\tilde{T})\\ \theta_1 = f_2 - f_3,\ \theta_2 = f_1 - f_3,\ \theta_3 = f_1 - f_3\end{gathered}$$

and describes the propagation of small perturbations along the front of a non-linear waveguide parallel to the straight line $c_x X + c_y Y = 0$. The perturbations φ_j change the curvature of the wavefront and the phase of resonantly interacting waves and propagate with their group velocities. The projections of their velocities onto the Y axis are equal to $\nabla_k^y \omega_j$.

Thus, we have proved the stability of the solutions considered with respect to slow changes in the boundaries of the waveguide and the amplitudes of the waves propagating along it.

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